IRREDUCIBLE REPRESENTATIONS OF THE QUANTUM WEYL ALGEBRA AT ROOTS OF UNITY GIVEN BY MATRICES

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Abstract. To describe the representation theory of the quantum Weyl algebra at an lth primitive root γ of unity, Boyette, Leyk, Plunkett, Sipe, and Talley found all nonsingular irreducible matrix solutions to the equation $yx - \gamma xy = 1$, assuming $yx \neq xy$. In this note, we complete their result by finding and classifying, up to equivalence, all irreducible matrix solutions (X, Y), where X is singular.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field, γ a nonzero scalar in $\mathbb{K}$. The irreducible representations of the quantized Weyl algebra $\mathbb{A} = \mathbb{K}\{x, y\}/\langle yx - \gamma xy - 1\rangle$ has been constructed and classified in [2] and [3]. It is well known that, when γ is a root of unity, any irreducible representation of $\mathbb{A}$ is finite dimensional. In [1], Boyette, Leyk, Plunkett, Sipe, and Talley present a linear algebra method to prove the Drozd-Guzner-Ovsienko result in the case when γ is a primitive root of unity. Let V be an $\mathbb{A}$-module and $\rho : \mathbb{A} \to \mathrm{End}(V) \hookrightarrow M_n(\mathbb{K})$ be a representation of $\mathbb{A}$. Then ρ is irreducible, i.e., V is simple, if and only if $\rho(\mathbb{A}) = \mathrm{M}_n(\mathbb{K})$. Their approach then is to find explicitly, up to equivalence, all irreducible matrix solutions to the equation $yx - \gamma xy = 1$ with $yx \neq xy$. Two $n \times n$ matrices X and Y form a *matrix solution* if $YX - \gamma XY = I$, the identity matrix. A solution (X, Y) is *irreducible* if every matrix in $M_n(\mathbb{K})$ can be written as a noncommutative polynomial in X and Y over $\mathbb{K}$, assuming the zero power of a matrix is the identity matrix. Two solutions (X, Y) and (M, N) are *equivalent* if there is a nonsingular matrix Q such that $QXQ^{-1} = M$ and $QYQ^{-1} = N$.

Throughout, γ is an lth primitive root of unity for some integer $l \geq 2$. Unless specified otherwise, any *solution* is a matrix solution to the equation $yx - \gamma xy = 1$.

1.1. **Nonsingular solutions.**

Suppose (X, Y) is an irreducible solution and $U = YX - XY \neq 0$. The following are proved in [1].

(i) $YX^l = X^lY$ and $Y^lX = XY^l$, and so X^l and Y^l are scalar matrices. It follows that any irreducible matrix solutions is at most $l \times l$.

2010 *Mathematics Subject Classification.* 16D60, 81R50.

Key words and phrases. quantum Weyl algebra, representations, matrix equations.

The first author was supported by an undergraduate research grant from CST at Southeastern Louisiana University. Research of the second author supported by the Louisiana Board of Regents [LEQSF(2012-15)-RD-A-20].

(ii) $UX = \gamma XU$, $YU = \gamma UY$, and U is nonsingular. If X has one nonzero eigenvalue, say λ, then X has at least l distinct eigenvalues, $\gamma\lambda, \gamma^2\lambda, \ldots, \gamma^l\lambda$. In particular, (X, Y) is at least $l \times l$.

(iii) When X has at least one nonzero eigenvalue, all irreducible solutions are $l \times l$ and found explicitly in [1] as follows.

$$X_\lambda = \lambda \begin{pmatrix} \gamma & 0 & \ldots & 0 \\ 0 & \gamma^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \gamma^l \end{pmatrix} \quad Y_{\lambda\, b's} = \begin{pmatrix} \frac{1}{(1-\gamma)\gamma\lambda} & b_1 & 0 & \ldots & 0 & 0 \\ 0 & \frac{1}{(1-\gamma)\gamma^2\lambda} & b_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{(1-\gamma)\gamma^{l-1}\lambda} & b_{l-1} \\ b_l & 0 & 0 & \ldots & 0 & \frac{1}{(1-\gamma)\gamma^l\lambda} \end{pmatrix}$$

where λ, b's are nonzero scalars in $\mathbb{K}$. The two matrices in each of these solutions are both nonsingular. These solutions are corresponding to the x-, y-torsion-free simple $\mathbb{A}$-modules. (cf. [3, Summary 3.7])

However, for an irreducible solution (X, Y), it is not always true that X has nonzero eigenvalues. For example, when $l = 2$, $\gamma = -1$, the matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form an irreducible solution to the equation $yx + xy = 1$. In fact, this solution gives the only simple A-module, where $\mathbb{A} = \mathbb{K}\{x, y\}/\langle yx + xy = 1\rangle$, that is both x- and y-torsion. (cf. [3, Example 4.1]) Hence, the remark (ii) in [1] and the assertion in [1, Proposition 2] are incorrect.

In this note, we complete [1, Proposition 2] by finding, up to equivalence, all irreducible matrix solutions (X, Y), where X only has zero eigenvalue. Moreover, the two types of irreducible matrix solutions, with X nonsingular or singular, are classified up to equivalence. These irreducible solutions give explicitly all irreducible representations of the quantum Weyl algebra at the root γ.

Acknowledgement. This problem arose in conversations between the second author and her Ph. D. advisor, E. Letzter. The authors are thankful for his generosity. The authors are also grateful to the referee for the comments that lead to Lemma 2.4.

2. Irreducible matrix solutions with X singular

Direct computation shows that the following is a solution.

2.1. **Solution.**

$$X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{l\times l} \qquad Y = Y_\beta = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \beta \\ \sum_{i=0}^{l-2}\gamma^i & 0 & \dots & 0 & 0 & 0 \\ 0 & \sum_{i=0}^{l-3}\gamma^i & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1+\gamma & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

where β is a scalar in $\mathbb{K}$. The following on the matrices X and Y are either standard or straightforward.

(i) Note that X is the well-known upper shift matrix, and $X^l = 0$. Fix a positive integer $u \leq l-1$. Then X^u is a matrix with uth superdiagonal line all ones and zeroes elsewhere. Premultiplying a matrix A by X^u results in a matrix whose last u rows are all zeroes and the first $l-u$ rows are the last $l-u$ rows of the matrix A.

(ii) Note that each $Y_{k,k-1}$ along the subdiagonal line of Y is the only nonzero entry on the kth row of Y. Fix a positive integer $v \leq l-1$. It follows that

$$(Y^v)_{l,k} = \begin{cases} 0 & k \neq l-v \\ \prod_{j=l-v+1}^{l}\sum_{i=0}^{l-j}\gamma^i & k = l-v \end{cases}$$

That is, the only nonzero entry on the lth row of the matrix Y^v is $(Y^v)_{l,l-v}$.

2.2. **Proposition.** *The solution (X,Y) in (2.1) is irreducible.*

Proof. Let $H = \mathrm{Span}_{\mathbb{K}}\{X^iY^j;\quad 0 \leq i,j \leq l-1\}$. For any positive integers m, $n \leq l$, we will show that the elementary matrices $\mathbf{e}_{mn}$ are in H.

For fixed integers m, $n \leq l$, consider the matrix $X^{l-m}Y^{l-n}$. By (2.1 i), the last nonzero row of $X^{l-m}Y^{l-n}$ is the mth row, which is the same as the lth row of the matrix Y^{l-n}. By (2.1 ii), the only nonzero entry in the lth row of Y^{l-n} is

$$(Y^{l-n})_{l,n} = \prod_{j=n+1}^{l}\sum_{i=0}^{l-j}\gamma^i$$

Hence,

$$X^{l-m}Y^{l-n} = \Big(\prod_{j=n+1}^{l}\sum_{i=0}^{l-j}\gamma^i\Big)\mathbf{e}_{mn} + \Big(\sum_{s=1}^{m-1}\sum_{t=1}^{l}(X^{l-m}Y^{l-n})_{s,t}\Big)\mathbf{e}_{st}$$

When $m=1$, we have

$$X^{l-1}Y^{l-n} = \Big(\prod_{j=n+1}^{l}\sum_{i=0}^{l-j}\gamma^i\Big)\mathbf{e}_{1n}$$

for $n = 1, \dots, l$. Then, by induction on m, it shows that any $\mathbf{e}_{mn} \in H$. $\square$

2.3. **Proposition.** *Suppose (A,B) with $BA \neq AB$ is an irreducible solution. If the only eigenvalue of A is 0, then (A,B) is equivalent to a solution as follows.*

$$X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}_{l\times l} \quad Y_{\alpha' s} = \begin{pmatrix} \gamma^{l-1}\alpha_l & \gamma^{l-2}\alpha_{l-1} & \dots & \gamma^2\alpha_3 & \gamma\alpha_2 & \alpha_1 \\ \sum_{i=0}^{l-2}\gamma^i & \gamma^{l-2}\alpha_l & \dots & \gamma^2\alpha_4 & \gamma\alpha_3 & \alpha_2 \\ 0 & \sum_{i=0}^{l-3}\gamma^i & \dots & \gamma^2\alpha_5 & \gamma\alpha_4 & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1+\gamma & \gamma\alpha_l & \alpha_{l-1} \\ 0 & 0 & \dots & 0 & 1 & \alpha_l \end{pmatrix}$$

where α_i are scalars in $\mathbb{K}$ for $i = 1, 2, \dots, l$.

Proof. Suppose that (A,B) with $BA \neq AB$ is an $n \times n$ irreducible solution for some positive integer n and that the only eigenvalue of A is 0. Let $C = QAQ^{-1}$ be the Jordan Canonical form for A, where Q is an $n \times n$ matrix. Set $D = QBQ^{-1}$. Then $(C,\, D)$ is also an irreducible solution.

Suppose the Jordan blocks J_i in C are $m_i \times m_i$ for $i = 1, \dots, k$. Partition the matrix D into blocks (D_{ij}), where D_{ij} are $m_i \times m_j$ matrices, for $1 \leq i, j \leq k$. Then we have $D_{ii}J_i - \gamma J_i D_{ii} = 1 \quad \text{for} \quad i = 1, 2, \dots, k$. Fix i, let $D_{ii} = (d_{pq})_{m_i \times m_i}$. Note that along the diagonal line of the matrix $D_{ii}J_i - \gamma J_i D_{ii}$ should be all ones, i.e.,

$$1 = 0 - \gamma d_{21},\ 1 = d_{21} - \gamma d_{32},\ \dots,\ 1 = d_{m_i-1\, m_i-2} - \gamma d_{m_i\, m_i-1},\ \text{and } 1 = d_{m_i\, m_i-1}.$$

It then follows that $1 + \gamma + \gamma^2 + \dots + \gamma^{m_i-1} = 0$. But γ is an lth primitive root of unity. Thus m_i must be an integer multiple of l. Hence, $n = \sum_i m_i$ must be greater than or equal to l. It then follows from (1.1, i) that C has to be the $l \times l$ Jordan block with zeroes on the diagonal line, i.e., $C = QAQ^{-1} = X$.

Now, it is sufficient to show that $D = QBQ^{-1}$ must have the form $Y_{\alpha'\mathrm{s}}$. It follows from $BA - \gamma AB = 1$ that $DX - \gamma XD = 1$. We will explore the problem element-wise. By (2.1, i), we have

$$(DX)_{ij} = \begin{cases} d_{i\,j-1}, & j > 1 \\ 0, & j = 1 \end{cases} \quad \text{and} \quad (\gamma XD)_{ij} = \begin{cases} \gamma d_{i+1\,j} & i < l \\ 0, & i = l \end{cases}$$

Since $i + 1 \neq 1$ and $j - 1 \neq l$, it is clear that d_{1l} is a free variable.

Case 1, $i < j$. Fix $1 \leq k \leq l-1$, the kth superdiagonal line of $DX - \gamma XD$ has entries

$$(DX - \gamma XD)_{i,i+k} = d_{i,i+k-1} - \gamma d_{i+1,i+k} \quad \text{for} \quad i = 1, 2, \dots, l-k.$$

That is, $d_{1,k-1} - \gamma d_{2\,k} = 0,\ d_{2\,k} - \gamma d_{3,k+1} = 0\ \dots\ d_{l-k,l-1} - \gamma d_{l-k+1,l} = 0$. Then, inductively,

$$d_{i,i+k-1} = \gamma^{l-k} d_{l-k+1,l} \quad \text{for} \quad i = 1, 2, \dots, l-k,$$

Therefore, on the lth column of D, we get free variables $d_{2\,l}, \dots, d_{l-1,l}$.

Case 2, $i = j$. We have

$$1 = (DX - \gamma XD)_{ii} = \begin{cases} 0 - \gamma d_{21} & i = 1 \\ d_{i,i-1} - \gamma d_{i+1,i} & 1 < i < l \\ d_{l,l-1} - 0 & i = l \end{cases}$$

Thus, the entries on the subdiagonal line of D are

$$d_{21} = -\frac{1}{\gamma} = \sum_{i=0}^{l-2} \gamma^i, \quad d_{32} = \frac{d_{21} - 1}{\gamma} = \sum_{i=0}^{l-3} \gamma^i, \quad \ldots, \quad d_{l-1,l-2} = 1 + \gamma, \text{ and } d_{l,l-1} = 1$$

Case 3, $i > j$. We have

$$0 = (DX - \gamma XD)_{ij} = \begin{cases} 0 - \gamma d_{i+1,1} & j = l \text{ and } j < i < l \\ d_{l,j-1} & i = l \text{ and } 1 < j < i \\ d_{i,j-1} - \gamma d_{i+1,j} & i < l \text{ and } 1 < j < i \end{cases}$$

It is not hard to see, inductively, that $d_{ij} = 0$ for any $2 < i \leq l$ and $1 \leq j \leq i - 2$. The proposition follows. □

2.4. **Lemma.** *Any solution* $(X, Y_{\alpha' s})$ *in* (2.3) *is equivalent to a solution* (X, Y_β) *in* (2.1). *The equivalence classes of solutions in* (2.3) *are* $[\,(X, Y_\beta)\,]$ *for* $\beta \in \mathbb{K}$.

Proof. Suppose (X, Y_β) is a solution as in (2.1). Direct computation shows that the following uppertriangular matrix P is a nonsingular matrix such that $P^{-1}XP = X$.

$$P = \begin{pmatrix} 1 & p_{l-1} & p_{l-2} & \ldots & p_2 & p_1 \\ & 1 & p_{l-1} & \ldots & p_3 & p_2 \\ & & & \ddots & & \\ & & & & 1 & p_{l-1} \\ & & & & & 1 \end{pmatrix}_{l \times l}$$

where p_k are scalars in $\mathbb{K}$ for $k = 1, \ldots, l-1$. Then the matrices X and $P^{-1}Y_\beta P$ also form a solution. It is shown in the proof of (2.3) that if (X, D) is a solution then D must be one of the form $Y_{\alpha'\text{s}}$. Thus, $P^{-1}Y_\beta P = Y_{\alpha'\text{s}}$ for some $\alpha_1, \ldots, \alpha_l$. Consider the matrices $PY_{\alpha'\text{s}}$ and $Y_\beta P$. The second row of $Y_\beta P$ is

$$\sum_{i=0}^{l-2} \gamma^i \cdot (1 \quad p_{l-1} \quad \ldots \quad p_1).$$

Set $\bar{Y}_{\alpha'\text{s}}$ be the lower right $(l-1) \times (l-1)$ block of the matrix $Y_{\alpha'\text{s}}$ and R be the $(l-1) \times l$ matrix obtained by removing the last row of the matrix X. Then the second row of $PY_{\alpha'\text{s}}$ can be written as

$$(0 \quad 1 \quad p_{l-1} \quad \ldots \quad p_2) \cdot Y_{\alpha'\text{s}} = \Big(\sum_{i=0}^{l-2} \gamma^i \quad 0 \quad \ldots \quad 0 \Big) + (1 \quad p_{l-1} \quad \ldots \quad p_2) \cdot \bar{Y}_{\alpha'\text{s}} R$$

It then follows from $PY_{\alpha'\text{s}} = Y_\beta P$ that

$$\sum_{i=0}^{l-2} \gamma^i \cdot \begin{pmatrix} p_{l-1} & \dots & p_1 \end{pmatrix} = \begin{pmatrix} 1 & p_{l-1} & \dots & p_2 \end{pmatrix} \cdot \bar{Y}_{\alpha'\text{s}}$$

Note that $\bar{Y}_{\alpha'\text{s}}$ is the sum of the upper triangular matrix

$$M = \begin{pmatrix} \gamma^{l-2}\alpha_l & \gamma^{l-3}\alpha_{l-1} & \dots & \gamma\alpha_3 & \alpha_2 \\ 0 & \gamma^{l-3}\alpha_l & \dots & \gamma\alpha_4 & \alpha_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \gamma\alpha_l & \alpha_{l-1} \\ 0 & 0 & \dots & 0 & \alpha_l \end{pmatrix}$$

and the $(l-1)\times(l-1)$ subdiagonal matrix L with entries $\sum_{i=0}^{l-3}\gamma^i, \quad \dots, \ 1+\gamma, \ 1$ along the diagonal line. Then we have

$$\sum_{i=0}^{l-2} \gamma^i \cdot \begin{pmatrix} 1 & p_{l-1} & \dots & p_1 \end{pmatrix} - \begin{pmatrix} 1 & p_{l-1} & \dots & p_2 \end{pmatrix} \cdot L = \begin{pmatrix} 1 & p_{l-1} & \dots & p_2 \end{pmatrix} \cdot M$$

and so

$$\left(\gamma^{l-2}\cdot p_{l-1} \quad \gamma^{l-3}(1+\gamma)\cdot p_{l-2} \quad \dots \quad \gamma\sum_{i=0}^{l-3}\gamma^i\cdot p_2 \quad \sum_{i=0}^{l-2}\gamma^i\cdot p_1\right) = (1 \ p_{l-1} \ \dots \ p_2)\cdot M$$

Therefore
(*)

$$\begin{cases} p_{l-1} = \alpha_l \\ p_k = \left(\sum_{i=0}^{l-1-k}\gamma^i\right)^{-1} \cdot \left(\alpha_{k+1} + \sum_{i=k+1}^{l-1}\alpha_{l+k+1-i}\cdot p_i\right) \quad \text{for} \quad k = l-2,\dots,1 \end{cases}$$

Moreover, it follows from $(PY_{\alpha'\text{s}})_{1,l} = (Y_\beta P)_{1,l}$ that

$$\beta = \alpha_1 + \sum_{i=1}^{l-1}\alpha_{l+1-i}\cdot p_i \tag{**}$$

This shows that any solution $(X, Y_{\alpha'\text{s}})$ is equivalent to a solution (X, Y_β) for some β in $\mathbb{K}$. The equivalence condition is given by a polynomial condition on α's and β that can be obtained inductively from (*) and (**). Simply arguing by determinant, we have the equivalence classes are $[\,(X, Y_\beta)\,]$ for $\beta \in \mathbb{K}$. □

The preceding lemma shows that, up to equivalence, the only irreducible solution with both determinants equal to zero is the solution (X, Y_0). This solution corresponds to the only x- and y-torsion simple module $L(0)$ over the quantum Weyl algebra $\mathbb{A}$ at the root γ. (cf. [3, Example 4.1]) Next, we provide the equivalence classification for irreducible solutions found in [1], which are corresponding to the x- and y-torsion-free simple $\mathbb{A}$-modules.

2.5. **Lemma.** *Any solution* $(X_\lambda, Y_{\lambda\, b's})$ *in* (1.1 iii) *is equivalent to a solution* $(X_\lambda, Y_{\lambda\,\eta})$ *where*

$$Y_{\lambda\,\eta} = \begin{pmatrix} \frac{1}{(1-\gamma)\gamma\lambda} & 1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{(1-\gamma)\gamma^2\lambda} & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{(1-\gamma)\gamma^{l-1}\lambda} & 1 \\ \eta & 0 & 0 & \dots & 0 & \frac{1}{(1-\gamma)\gamma^l\lambda} \end{pmatrix}$$

for some $\eta \in \mathbb{K}^\times$*. The equivalence classes of solutions in* (1.1 iii) *are* $[\,(X_\lambda, Y_{\lambda\,\eta})\,]$ *where* $\eta \in \mathbb{K}^\times$ *and* $\lambda \in \mathbb{K}^\times/\langle\gamma\rangle$*.*

Proof. Let $(X_\lambda, Y_{\lambda\, b's})$ and $(X_{\lambda'}, Y_{\lambda'\, c's})$ be two irreducible solutions as in (1.1 iii). They are equivalent only if $\lambda'^l = \lambda^l$ and $\prod_{i=1}^l b_i = \prod_{i=1}^l c_i$. Suppose $\lambda' = \gamma^i\lambda$ for some $1 \le i \le l$. Then, by using an appropriate permutation matrix, we have $(X_{\gamma^i\lambda},\ Y_{\gamma^i\lambda\, b's})$ is equivalent to $(X_\lambda,\ Y_{\lambda\, b's})$, where the entries b's in $Y_{\gamma^i\lambda\, b's}$ and those in $Y_{\lambda\, b's}$ are the same, up to the corresponding permutation. It then remains to show that $(X_\lambda,\ Y_{\lambda\, b's})$ is equivalent to $(X_\lambda,\ Y_{\lambda\,\eta})$, where $\eta = \prod_{i=1}^l b_i$. This can be done by using a diagonal matrix P with entries

$$1,\ b_1,\ b_1b_2,\ \dots,\ b_1\cdots b_{l-1}$$

along the diagonal line. □

Combining Proposition 2.3, Lemma 2.4, (1.1 iii) and Lemma 2.5, we have

2.6. **Proposition.** ([2, Theorem 5.8]) *Any irreducible solution* (A, B) *to the equation* $yx - \gamma xy = 1$*, in which* $BA \neq AB$*, is equivalent to either the solution* $(X_\lambda, Y_{\lambda\,\eta})$ *in* (2.5) *or the solution* (X, Y_β) *in* (2.1)*.*

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